Final Review Day 1, Math 215, Spring11, section 2

For full credit, show all work.

1 Consider the linear system

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

For what values of k does the system have a unique solution? No solution? Infinitely many solutions?

2 Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points in the line $x_2 = x_1$ and then reflects the result in the horizontal x_1-axis.

3 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix}$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the columns of the 3×3 identity matrix. Determine if T is a one-to-one linear transformation. Explain.

4 Suppose M is a 4×4 matrix with linearly independent columns, explain why the equation $M\mathbf{x} = \mathbf{b}$ has a unique solution for *every* $\mathbf{b} \in \mathbb{R}^4$.

5 Let $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $B' = \{\mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1 + 3\mathbf{e}_2\}$ are bases for $\mathbb{R}^2$. Find the change of basis matrix from B to B'.

6 Let B and B' as defined in problem 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\mathbf{e}_1) = 3\mathbf{e}_1 - 2\mathbf{e}_2$, $T(\mathbf{e}_2) = \mathbf{e}_1 + 4\mathbf{e}_2$. Write down the matrix representation for T in the standard basis B. Then use the change of basis matrix from problem 5 to find the matrix representation for T in basis B'.

7 Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. Use a) Gaussian elimination, b) cofactor expansion and c) permutation, to compute its determinant.

8 Let $V = \mathcal{D}$ is the vector space of differentiable functions and $W = \{f \in \mathcal{D} | f'(x) + f(x) = 0\}$. Show that W is a subspace of V.

9 $W = \left\{ A \in M_{22} | A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \right\}$ is a subspace of M_{22}. Find a basis for W.

10 Find the coordinate vector for $p(x) = 2x^2 + x + 3$ in the basis $\mathcal{B} = \{1, x - 1, (x - 1)^2\}$ for $\mathcal{P}_2$.

11 Let $T : P_2 \to P_2$ where $T(1) = x$, $T(x - 1) = x^2 + 1$, $T((x - 1)^2) = 1 - x$. Find $T(2x^2 + x + 3)$.

12 Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. Find its eigenvalues and a basis for its eigenspaces. If A is diagonalizable, find the invertible matrix P and diagonal matrix D such that $D = P^{-1}AP$.

13 Use the diagonalization from the previous problem to solve the system of differential equation $\mathbf{x}' = A\mathbf{x}$ where A is as defined above, with initial condition $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

14 Find the eigenvalues of the matrix $A = \begin{pmatrix} 11 & -15 \\ 6 & -7 \end{pmatrix}$. If this matrix describes a dynamical system $x_n = Ax_{n-1}$, what type of fixed point is the zero vector?

15 Every year, 10% of all University of Okoboji students change their major to mathematics, and 25% of math majors change their majors to something else, or graduate. If the university enrollment remains a constant 13,500 and the math department starts with 100 majors, what is the long term departmental enrollment?

16 Find the orthogonal projection of $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ onto the plane in $\mathbb{R}^3$ whose equation is $x - 2y + z = 0$.

17 Find the orthogonal decomposition of $\mathbf{x} = \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ with respect to the subspace $W = \text{span}\left(\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} \right)$.

18 Let $\mathbf{v} = \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_1 = \frac{1}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{9}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ and $W = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$. Let $U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$. Compute $(UU^T)\mathbf{v}$. What is the geometric interpretation of that?

19 True/False. Give reasons.

i) In some cases, it is possible for 6 vectors to span $\mathbb{R}^5$.

ii) If a matrix A is $m \times n$ and if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for some $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$.

iii) If a system of linear equations has two different solutions, then it has infinitely many solutions.

iv) If $\mathbf{v}_1$ and $\mathbf{v}_2$ span a plan in $\mathbb{R}^3$ and if $\mathbf{v}_3$ is not in that plane, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

v) The rank of a matrix is equal to the number of its nonzero rows.

vi) If A is an $m \times n$ matrix of rank m, then the system $A\mathbf{x} = \mathbf{b}$ must have a solution.

vii) If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\det(A) = 0$.

viii) If A, B are $n \times n$ matrices and B is invertible, then $\det(B^{-1}AB^T) = \det(A)$.

ix) The sum of two eigenvectors of a matrix is always an eigenvector of the matrix.

x) If λ is an eigenvalue of A, then the geometric multiplicity of λ equals the rank of $A - \lambda I$.

xi) Every vector space has a finite basis.

xii) If $T : V \to W$ is a function from a vector space V to a vector space W, then T is linear if and only if $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

xiii) If T is a linear transformation, then T maps a linearly independent set to a linearly independent set.

xiv) The empty set is a subspace of every vector space.

xv) A matrix Q is orthogonal if and only if $\det(Q) = \pm 1$.

xvi) If A is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = m$.

xvii) Any orthogonal set of vectors is linearly independent. What if "orthogonal" is replaced by "orthonormal"?