Final Review Day 1, Math 215, Spring11, section 2

For full credit, show all work.

1 Consider the linear system

\[ \begin{align*}
    kx + y + z &= 1 \\
    x + ky + z &= 1 \\
    x + y + kz &= 1
\end{align*} \]

For what values of \( k \) does the system have a unique solution? No solution? Infinitely many solutions?

2 Find the standard matrix of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) that reflects points in the line \( x_2 = x_1 \) and then reflects the result in the horizontal \( x_1 \)-axis.

3 Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation such that \( T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, T(e_2) = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, T(e_3) = \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix} \), where \( e_1, e_2, e_3 \) are the columns of the \( 3 \times 3 \) identity matrix. Determine if \( T \) is a one-to-one linear transformation. Explain.

4 Suppose \( M \) is a \( 4 \times 4 \) matrix with linearly independent columns, explain why the equation \( Mx = b \) has a unique solution for \( \forall b \in \mathbb{R}^4 \).

5 Let \( B = \{ e_1, e_2 \} \) and \( B' = \{ e_1 + e_2, 2e_1 + 3e_2 \} \) are bases for \( \mathbb{R}^2 \). Find the change of basis matrix from \( B \) to \( B' \).

6 Let \( B \) and \( B' \) as defined in problem 5. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( T(e_1) = 3e_1 - 2e_2 \), \( T(e_2) = e_1 + 4e_2 \). Write down the matrix representation for \( T \) in the standard basis \( B \). Then use the change of basis matrix from problem 5 to find the matrix representation for \( T \) in basis \( B' \).

7 Let \( A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \). Use a) Gaussian elimination, b) cofactor expansion and c) permutation, to compute its determinant.

8 Let \( V = \mathcal{D} \) is the vector space of differentiable functions and \( W = \{ f \in \mathcal{D} | f'(x) + f(x) = 0 \} \). Show that \( W \) is a subspace of \( V \).

9 \( W = \left\{ A \in M_{22} | A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \right\} \) is a subspace of \( M_{22} \). Find a basis for \( W \).

10 Find the coordinate vector for \( p(x) = 2x^2 + x + 3 \) in the basis \( B = \{ 1, x - 1, (x - 1)^2 \} \) for \( \mathcal{P}_2 \).

11 Let \( T : \mathcal{P}_2 \to \mathcal{P}_2 \) where \( T(1) = x, T(x - 1) = x^2 + 1, T((x - 1)^2) = 1 - x \). Find \( T(2x^2 + x + 3) \).

12 Let \( A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \). Find its eigenvalues and a basis for its eigenspaces. If \( A \) is diagonalizable, find the invertible matrix \( P \) and diagonal matrix \( D \) such that \( D = P^{-1}AP \).
13 Use the diagonalization from the previous problem to solve the system of differential equation
\[ \mathbf{x}' = A\mathbf{x} \] where \( A \) is as defined above, with initial condition \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

14 Find the eigenvalues of the matrix \( A = \begin{pmatrix} 11 & -15 \\ 6 & -7 \end{pmatrix} \). If this matrix describes a dynamical system \( x_n = Ax_{n-1} \), what type of fixed point is the zero vector?

15 Every year, 10% of all University of Okoboji students change their major to mathematics, and 25% of math majors change their majors to something else, or graduate. If the university enrollment remains a constant 13,500 and the math department starts with 100 majors, what is the long term departmental enrollment?

16 Find the orthogonal projection of \( \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \) onto the plane in \( \mathbb{R}^3 \) whose equation is \( x - 2y + z = 0 \).

17 Find the orthogonal decomposition of \( \mathbf{x} = \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix} \) with respect to the subspace \( W = \text{span} \left( \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \right) \).

18 Let \( \mathbf{v} = \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \) and \( W = \text{span}(\mathbf{u}_1, \mathbf{u}_2) \). Let \( U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \). Compute \( (UU^T)v \). What is the geometric interpretation of that?

19 True/False. Give reasons.

i) In some cases, it is possible for 6 vectors to span \( \mathbb{R}^5 \).

ii) If a matrix \( A \) is \( m \times n \) and if the equation \( A\mathbf{x} = \mathbf{b} \) has a solution for some \( \mathbf{b} \), then the columns of \( A \) span \( \mathbb{R}^m \).

iii) If a system of linear equations has two different solutions, then it has infinitely many solutions.

iv) If \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) span a plane in \( \mathbb{R}^3 \) and if \( \mathbf{v}_3 \) is not in that plane, then \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a linearly independent set.

v) The rank of a matrix is equal to the number of its nonzero rows.

vi) If \( A \) is an \( m \times n \) matrix of rank \( m \), then the system \( A\mathbf{x} = \mathbf{b} \) must have a solution.

vii) If \( A \) is an \( n \times n \) matrix and \( A\mathbf{x} = \mathbf{0} \) for some \( \mathbf{x} \neq \mathbf{0} \), then \( \det(A) = 0 \).

viii) If \( A, B \) are \( n \times n \) matrices and \( B \) is invertible, then \( \det(B^{-1}AB^T) = \det(A) \).

ix) The sum of two eigenvectors of a matrix is always an eigenvector of the matrix.

x) If \( \lambda \) is an eigenvalue of \( A \), then the geometric multiplicity of \( \lambda \) equals the rank of \( A - \lambda I \).
xi) Every vector space has a finite basis.

xii) If \( T : V \to W \) is a function from a vector space \( V \) to a vector space \( W \), then \( T \) is linear if and only if \( T(x + y) = T(x) + T(y) \).

xiii) If \( T \) is a linear transformation, then \( T \) maps a linearly independent set to a linearly independent set.

xiv) The empty set is a subspace of every vector space.

xv) A matrix \( Q \) is orthogonal if and only if \( \det(Q) = \pm 1 \).

xvi) If \( A \) is an \( m \times n \) matrix, then \( \text{rank}(A) + \text{nullity}(A) = m \).

xvii) Any orthogonal set of vectors is linearly independent. What if “orthogonal” is replaced by “orthonormal”? 