Homework 14

1 Let V be a vector space and $B = \{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\}$ be a basis for V and $\vec{v}, \vec{v}_1, \cdots, \vec{v}_k \in V$. If $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ with weights $c_1, \cdots, c_k$, show that the same relation holds for the associated coordinate vectors, i.e. $[\vec{v}]_B$ is a linear combination of $[\vec{v}_1]_B, [\vec{v}_2]_B, \cdots, [\vec{v}_k]_B$ with the same weights.

2 Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

3 Show that an $n \times n$ matrix can have at most n distinct eigenvalues.

4 Suppose an $n \times n$ matrix A has eigenvalue λ. Show that the λ-eigenspace is a subspace of $\mathbb{R}^n$.

5a) Determine the eigenvalue(s) of $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (without using *characteristic equation* that we have not discussed) and find a basis for the associated eigenspace.

b) The **algebraic multiplicity** of an eigenvalue λ is the number of copies of λ as an eigenvalue for A (at this point, I just want you to read off from A how many times a number appears as an eigenvalue of A). The **geometric multiplicity** of λ is the dimension of the eigenspace, i.e. the number of linearly independent eigenvectors associated with λ. What is the algebraic and geometric multiplicity for the eigenvalue(s) you obtained for A in part (a)?

c) Find $\vec{v}_2$ such that $(A - \lambda I)\vec{v}_2 = \vec{v}_1$ where $\vec{v}_1$ is an eigenvector for λ that you obtained in part (a). $\vec{v}_2$ is called a *generalized eigenvector*. Are $\vec{v}_1, \vec{v}_2$ linearly independent? Explain.

Note: This exercise is to demonstrated that if we do not have enough eigenvectors to form a basis for the underlying vector space because the algebraic multiplicity > geometric multiplicity for eigenvalue λ, we can complete the set of linearly independent eigenvectors by generalized eigenvectors to form a basis for the underlying vector space.