**Homework 15 – cover lectures through 4/8**

1 The trace of an $n \times n$ matrix $A$ is the sum of its diagonal entries, i.e. $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$. Consider a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\text{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1 \lambda_2$, where $\lambda_1, \lambda_2$ are eigenvalues of $A$.

*Note: this relation between trace, determinant of $A$ and its eigenvalues persists for matrices of higher dimensions.*

2 $A$ is an $n \times n$ diagonalizable matrix. Show that $A$ satisfies its own characteristic equation by following these steps:

a) The characteristic polynomial, $f(\lambda)$, is an $n$ degree polynomial in $\lambda$, i.e. $f(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$. Consider $f(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n$. Find the corresponding polynomial for $D$ where $D$ is the diagonalization of $A$.

b) Show that the polynomial of $D$ from part (a) has to equal zero (**warning: do not already assume** $f(A) = 0$). Use this to conclude that $f(A) = 0$ and hence $A$ satisfies its own characteristic equation.

3 Let $A = \begin{bmatrix} 7 & -9 \\ 4 & -5 \end{bmatrix}$.

a) Argue that $A$ is not diagonalizable using the algebraic and geometric multiplicities of its eigenvalue(s). Find a basis for its eigenspace(s).

b) Recall from homework 14 that a vector $\vec{v}_2$ is a generalized eigenvector for an eigenvalue $\lambda$ if $(A - \lambda I)\vec{v}_2 = \vec{v}_1$ where $\vec{v}_1$ is an eigenvector associated with $\lambda$. Note that a matrix is not diagonalizable if there is no eigen-basis. We can complete the set of linearly independent eigenvectors by generalized eigenvectors to form a basis for the underlying vector space. Construct such a basis for $A$.

c) Use the basis from part (b) to construct the similarity transformation matrix $P$. Compute the similarity transformation $J = P^{-1}AP$. $J$ is called the *Jordan (canonical) form* of $A$, and is the simplest similar matrix of $A$.