Some Incomplete Proofs

Instead of supplying you full proofs that some of you are going to memorize rather than learning the ideas and writing of proofs, I will provide hints to guide your proof. You are welcome to email me your proofs for me to check.

1 Show that $c\mathbf{0} = \mathbf{0}$. You are only to use the algebraic properties listed on page 32 and nothing more.

The only properties of the zero vector is as an additive identity i.e. $\mathbf{0}+\mathbf{u} = \mathbf{u}$ and in the existence of an additive inverse, i.e. $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$. Note this has nothing to scalar multiplication that we are trying to show.

Consider $c\mathbf{0}+c\mathbf{u} = c(?) =$. Fill in the blanks and use the property you use. Then use the property of addiditive inverse. Add $-c\mathbf{u}$ to both sides of the equation. This should pops $\mathbf{0}$ into the picture.

2 Show that $0\mathbf{u} = \mathbf{0}$.

Start with $0\mathbf{u} + 0\mathbf{u} = (0 + 0)\mathbf{u}$ by property (?). What is the number $0 + 0$? Then use additive inverse by adding $-0\mathbf{u}$ to both sides of the equation. SImplify.

3 This is homework problem: Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

I will write this in details, since you have attempted in your homework.

Start by writing down what you need to show:

We want to show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. This means we need to find at least one set of c_1, c_2, c_3, c_4 not all zeros such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, by definition, there exists scalars d_1, d_2, d_3 not all zeros such that $d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 = \mathbf{0}$.

Choose $c_1 = d_1, c_2 = d_2, c_3 = d_3, c_4 = 0$. This is a set of scalars that are not all zeros. We shall show that this set of scalars satisfies the desired equation: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3) + 0\mathbf{v}_4 = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

Since we can find a set of scalars not all zeros such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$, by definition, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

4 If $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is linearly independent and $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$, with $c_1 \neq 0$, show that $\{\mathbf{u}, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is linearly independent.

Again, write down the meaning of what you need to show: If $\{\mathbf{u}, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is linearly independent, then there exists scalars $x_1, x_2, \cdots, x_k$ such that $x_1\mathbf{u} + x_2\mathbf{v_2} + \cdots + x_k\mathbf{v}_k = 0$.

Now, plug in $\mathbf{u}$ as a linear combination as given. You should now have an equation relating $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. Collect coefficients of each vector. Then use the fact that this is a linearly independent set to argue that each coefficient has to be zero. This should given you a set of equations for the scalars. You can solve them recursively to conclude $x_1, \cdots, x_k$ must be all zeros, one at a time.

5 Show that if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is a linearly dependent set, then one of the vectors can be written as a linear combination of the rest.

6 Suppose a linear transformation T maps a non-zero vector to the zero vector. Show that the linear transformation is not one-to-one.

What vector do you know that any linear transformation must map onto the zero vector? Think about how this will help you answer the question.

7 Suppose a linear transformation T only maps the zero vector to the zero vector. Show that T is one-to-one.

The proof is by contradiction:
Suppose T is not one-to-one. Then there are $\mathbf{u} \neq \mathbf{v}$ such that $T(\mathbf{u}) = T(\mathbf{v})$.

Then $T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$. By linearity $T(\mathbf{u}) - T(\mathbf{0}) = T(\mathbf{u} - \mathbf{v})$. Hence, $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. Since T only maps the zero vector to the zero vector, we must have $\mathbf{u} - \mathbf{v} = 0 \Rightarrow \mathbf{u} = \mathbf{v}$.

This shows it is not possible to find two different vectors that T maps to the same image. Hence, T is one-to-one.