Review for Test 3

This test covers sections 4.4–4.5, 5.1–part of 5.4, 5.6–5.7, 6.1–6.4. As before, prior chapters though not explicitly tested are still essential as tools for these latter sections. As always, lecture notes, assignments, written homework, quizzes are all relevant, *even if* they are not in the text. The textbook is only part of the resources.

Test format will be similar to previous tests, and possibly includes definitions, computations, true/false with justification and proofs.

The main point of the remainder of chapter 4 involves mapping from a general vector space to \( \mathbb{R}^n \) by the isomorphism given by the coordinate mapping determined by a chosen basis. 4.5 supplies the concept of dimension of a subspace/ vector space that gives an alternative way to check for a basis.

- Important definitions: basis, coordinate mapping, dimension of a vector space, change of coordinate matrix.

- Important skills (partial list):
  - Find coordinate vector for a vector relative to a basis
  - Find change of basis matrix
  - Find matrix representation of a linear transformation in general vector space under the coordinate mapping (this material and exercises are oddly placed in part of section 5.4)
  - Find dimension of \( \text{Col}(A), \text{Nul}(A) \) and other subspaces.
  - Know how to map a general vector space onto \( \mathbb{R}^n \) to solve problems, e.g. linear combination, linear independence, existence and uniqueness of solutions to linear systems etc.

Chapter 5 introduces the concepts of eigen-system and gives a glimpse of the power of these concepts in solving a vast area of application problems. Eigenvectors are special directions where the effect of the associated matrix transformation is simply a scaling by a number called eigenvalue. Eigenvectors are also where the effect of the matrix transformation is the most exaggerated. A linear system \( \vec{y} = A\vec{x} \) where \( \vec{x}, \vec{y} \) are defined as appropriate to respective application problems, can be studied via understanding eigenvalues and eigenvectors. We restrict our attention to real systems with real eigenvalues and eigenvectors.

- Important definitions: eigenvalue, eigenvector, eigen-space, eigen-basis, similar matrices, diagonalizable.

- Important skills (partial list):
  - Determine if a number/ vector is an eigenvalue/ eigenvector of a matrix.
  - Find the characteristic polynomial/ equation and eigenvalues of a matrix.
  - Find eigenvalues of a triangular matrix.
  - Algebraic/ geometric multiplicities of an eigenvalue.
  - Find a basis for an eigen-space.
– Diagonalizes a matrix $A$.
– Compute high powers of a diagonalizable matrix.
– Verify statements involving similarity transformation/ similar matrices.
– Solving discrete and continuous linear dynamical sytems.
– Specifying a fixed point is attracting, repelling or a saddle point

Chapter 6 ventures into orthogonality relations, which requires the structure of an inner product, which induces a norm which in turns induces a metric/ distance. Each of these operators have useful repercussion in geometric descriptions of vectors. One of the main points of orthogonality is orthogonal projections, that gives the best approximation of a vector on a subspace. This gives the natural notion of decomposing a vector into a sum of its orthogonal projections onto a subspace $W$ and its orthogonal complement $W^\perp$. When a basis is orthogonal, it allows us a convenient description of the orthogonal projection of a vector onto a subspace $W$. This leads to the construction of an orthogonal/ orthonormal basis using Gram-Schmidt.

• Important definitions: inner product, dot product, norm, distance/ metric, orthogonal/ orthonormal vectors/ set, orthogonal/ orthonormal basis, orthogonal projection, orthogonal complement, orthogonal decomposition, orthogonal matrix.

• Important skills (partial list):
  – Know the properties of inner product, norm, metric and how to verify them.
  – Compute length of a vector, distance between two vectors, angle between two vectors.
  – Normalize a vector.
  – Check a set for orthogonality directly or using a matrix by computing $A^T A$ where $A$ contains the set of vectors in question.
  – Compute the orthogonal projection of a vector onto another vector or on a subspace.
  – Decompose a vector into a component in a subspace $W$ and its orthogonal complement $W^\perp$.
  – Use orthogonal projections to obtain estimates on a subspace.
  – Know the special orthogonal complements involving column and null space of a matrix and how to describe those subspaces, e.g. in vector form, by specifying a basis etc.
  – Describe any vector in a subspace with an orthogonal basis in terms of a linear combination of the basis vectors.
  – Generate an orthogonal basis from any given basis.