Symmetric $N$-Soliton Solutions and Asymptotic Forms

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Abstract

The inverse scattering method for the KdV equation by the dressing method is presented. We derive symmetric $N$-soliton solutions and their asymptotic forms with equidistant eigenvalues.

Key words: the KdV equation, Inverse Scattering, the dressing method, the $\bar{\partial}$-problem, $N$-soliton solutions

1. Introduction

In the paper, we revisit the KdV equation that is possibly the most well-studied integrable nonlinear equation. $N$-soliton solutions for the KdV equation were first discovered by Gardner, Greene, Kruskal, and Miura, by the method which today we call the classical inverse scattering method. The KdV equation is also known as the complete integrable system [1]. While the KdV has been studied extensively, deriving periodic solutions of the KdV from $N$-soliton solutions seems yet unsolved. To obtain periodic solutions for integrable systems seem inevitably to require the field of algebraic differential geometry, because of their spectral property that is well treated in Riemann surfaces. Our purpose is to make an interesting observation of $N$-soliton solutions that could possibly lead to the KdV periodic solution with one finite forbidden band from $N$-soliton solution as a limit $N \to \infty$.

The technique of the $\bar{\partial}$-dressing method is described in the papers [2][3] in much more detail, and it must be noted that the $\bar{\partial}$-dressing method covered in our paper is not original, and rather the simplest case. We consider the KdV equation with the fast decaying potential, and the continuous spectrum is ignored for the purpose to study $N$-soliton solutions only. We introduce symmetric $\bar{N}$-soliton solutions that can be derived from the determinant formula [4] (we call the Hirota’s determinant in our paper) for $N$-soliton solutions. Among various spacing of eigenvalues in a finite interval, our interest is particularly in equidistant eigenvalues. We derive an asymptotic form of symmetric $N$-soliton solutions with equidistant eigenvalues for
a sufficiently large $N$. Numerical study in the last section illuminates a possible connection between limits of symmetric $N$-soliton solutions and the the cnoidal wave solutions.

2. Preliminary: the $\bar{\partial}$-Dressing Method

We consider a function $M(\lambda, \bar{\lambda})$ defined on the complex plain $\mathbb{C}$. The $\bar{\partial}$-derivative is given by

$$\bar{\partial} \equiv \frac{\partial}{\partial \lambda} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad x, y \in \mathbb{R}. \quad (1)$$

If a function $M(\lambda, \bar{\lambda})$ is analytic in the entire complex plane, then $\bar{\partial}M = 0$. We consider some function $M(\lambda, \bar{\lambda})$ not necessarily analytic:

$$\bar{\partial}M(\lambda, \bar{\lambda}) = f(\lambda, \bar{\lambda}) \quad (2)$$

yielding the $\bar{\partial}$-problem.

Solving the equation requires inverting the $\bar{\partial}$-derivative. To do that, We should find $\bar{\partial}$ derivative of rational function $\frac{1}{\lambda}$. A rational function $1/\lambda$ can be written as

$$\frac{1}{\lambda} = \lim_{\epsilon \to 0} \frac{\bar{\lambda}}{\lambda \bar{\lambda} + \epsilon^2}, \quad (3)$$

and its derivative is by a limiting procedure of $\epsilon$

$$\frac{\partial}{\partial \lambda} \left( \frac{1}{\lambda} \right) = \lim_{\epsilon \to 0} \frac{\epsilon^2}{(\lambda \bar{\lambda} + \epsilon^2)^2} = c\delta(\lambda), \quad (4)$$

where $c$ is some constant.

A constant $c$ can be easily determined by integrating the equation (4). Since $\delta(\lambda) = \delta(\lambda_R)\delta(\lambda_I)$ with $\lambda = \lambda_R + i\lambda_I$, it is two dimensional delta function. Thereby

$$c = \lim_{\epsilon \to 0} \int_0^{2\pi} \int_0^\infty \frac{\epsilon^2}{(r^2 + \epsilon^2)^2} rdrd\theta = \pi \quad (5)$$

where $d\lambda_Rd\lambda_I = rdrd\theta$ with $\lambda = re^{i\theta}$.

Then it can be easily shown that
\[
\frac{\partial}{\partial \lambda} \left( \frac{1}{\lambda - \xi} \right) = \pi \delta(\lambda - \xi), \quad \lambda, \xi \in \mathbb{C}.
\]  

(6)

We can rewrite the equation (2) by

\[
\overline{\partial} M(\lambda, \bar{\lambda}) = \frac{1}{2\pi i} \frac{\partial}{\partial \lambda} \int \int_D f(\xi, \bar{\xi}) \frac{d\xi \wedge d\bar{\xi}}{\xi - \lambda},
\]

(7)

where \( \int \equiv \int_{-\infty}^{\infty} \), and \( d\xi \wedge d\bar{\xi} \) satisfies skew-symmetry, \( d\xi \wedge d\bar{\xi} = -d\bar{\xi} \wedge d\xi \), and can be written as \( d\xi \wedge d\bar{\xi} = -2id\xi R d\xi I \). The domain \( D \) is taken to be the entire complex plane.

Integrating the equation (7) with respect to \( \lambda \) gives

\[
M(\lambda, \bar{\lambda}) = M_0(\lambda) + \frac{1}{2\pi i} \int \int_D f(\xi, \bar{\xi}) \frac{d\xi \wedge d\bar{\xi}}{\xi - \lambda}.
\]

(8)

If we let \( M_0(\lambda) = \frac{1}{2\pi i} \int \frac{M(\lambda, \bar{\lambda})}{\xi - \lambda} d\xi \), the equation (8) is known to be the generalized Cauchy formula.

Normalizing the function \( M(\lambda, \bar{\lambda}) \to 1 \) as \( |\lambda| \to \infty \) gives

\[
M(\lambda, \bar{\lambda}) = 1 + \frac{1}{2\pi i} \int \int_D f(\xi, \bar{\xi}) \frac{d\xi \wedge d\bar{\xi}}{\xi - \lambda}.
\]

(9)

We consider the non-local \( \overline{\partial} \)-problem as follows

\[
\overline{\partial} M(\lambda, \bar{\lambda}) = -\frac{1}{2i} \int \int_D d\eta \wedge d\bar{\eta} M(\eta, \bar{\eta}) T(\eta, \bar{\eta}, \lambda, \bar{\lambda}).
\]

(10)

We include a constant \(-1/2i\) for convenience in later formulation. One can solve the equation (10) by the same way and obtain an integral equation where \( M(\lambda, \bar{\lambda}) \) is normalized to 1 as \( \lambda \to \infty \)

\[
M(\lambda, \bar{\lambda}) = 1 + \frac{1}{4\pi} \int \int_D d\xi \wedge d\bar{\xi} \int \int_D d\eta \wedge d\bar{\eta} M(\eta, \bar{\eta}) T(\eta, \bar{\eta}, \xi, \bar{\xi})
\]

(11)

with the degenerate kernel \( T \).

The nonlocal \( \overline{\partial} \) problem with \( \delta \)-functional kernels corresponds to soliton solutions, and points, at which delta functions are nonzero, appear to be eigenvalues (discrete spectrum). In the paper, We consider the KdV equation

\[
u_t + 6uu_x + u_{xxx} = 0.
\]

(12)
3. The Degenerate Kernel $T$ and Algebraic Equations

We express the $\delta$-functional kernel $T$ by

$$T(\eta, \eta, \xi, \overline{\xi}) = e^{\phi(\eta)}T_0(\xi, \overline{\xi})e^{-\phi(\xi)}\delta(\eta + \xi)$$  \hfill (13)

with

$$T_0(\xi, \overline{\xi}) = \pi \sum_{k=1}^{N} h_k^2 \delta(\xi - \lambda_k),$$  \hfill (14)

and

$$\phi(\lambda) = \lambda x - 4\lambda^3 t,$$  \hfill (15)

where $h_k$ is some constant $\in \mathbb{C}$.

The continuous spectrum that is ignored in the formulation can be easily implemented via the kernel $T$ with

$$T_0(\xi, \overline{\xi}) = \frac{1}{2} \delta(\xi R) \rho(\xi) + \pi \sum_{k=1}^{N} h_k^2 \delta(\xi - \lambda_k) \delta(\xi + \lambda_k) e^{\phi(\xi)} - e^{-\phi(\xi)} \delta(\xi - \lambda_k).$$  \hfill (16)

The second term $\rho(\xi)$ is the jump along the imaginary axis of a function $M(\lambda, \overline{\lambda})$. It can be simply shown that $M^+ - M^- = \rho(\xi)$, where $M^+$ is analytic in the right-half of the complex plane, and $M^-$ analytic in the left-half of the complex plane. From this kernel, one can derive the Gelfand-Levitan equation to take into account for the dispersive waves.

It must be noted that we ignore the continuous spectrum from now on.

Since $\int \int f(\eta, \overline{\eta})\delta(\eta - \lambda)d\eta \wedge d\overline{\eta} = -2if(\lambda, \overline{\lambda})$, the equation (11) with the kernel (13) becomes

$$M(\lambda, \overline{\lambda}) = 1 + \frac{1}{2i} \sum_{k=1}^{N} \int \int \frac{1}{\xi - \lambda} M(-\xi, -\overline{\xi}) h_k^2 \delta(\xi - \lambda_k) e^{\phi(-\xi) - \phi(\xi)} d\xi \wedge d\overline{\xi}$$  \hfill (17)

$$= 1 + \sum_{k=1}^{N} \frac{1}{\lambda - \lambda_k} M(-\lambda_k, \overline{\lambda_k}) h_k^2 e^{\phi(-\lambda_k) - \phi(\lambda_k)}$$  \hfill (18)

$$= 1 + \sum_{k=1}^{N} \frac{f_k(x, t)}{\lambda - \lambda_k},$$  \hfill (19)
where \( f_k(x,t) = M(-\lambda_k, \bar{\lambda}_k) h_k^2 e^{-\lambda_k x + 8\lambda_k^2 t} \).

Then, now we have

\[
M(\lambda, \bar{\lambda}) = 1 + \sum_{n=1}^{N} \frac{f_n(x,t)}{\lambda - \lambda_n}
\]  

(20)

or

\[
M(-\lambda, -\bar{\lambda}) = 1 - \sum_{n=1}^{N} \frac{f_n(x,t)}{\lambda + \lambda_n}.
\]

(21)

Letting \( \lambda = \lambda_k \) gives

\[
M(-\lambda_k, -\bar{\lambda}_k) = 1 - \sum_{n=1}^{N} \frac{f_n(x,t)}{\lambda_k + \lambda_n}.
\]

(22)

By introducing \( g_n(x,t) = f_n(x,t)e^{\phi(\lambda_n)} \), we can derive the linear system of equations from the above equation \((g_k \equiv g_k(x,t), \phi_k \equiv \phi(\lambda_k))\)

\[
g_k + h_k^2 \sum_{n=1}^{N} \frac{g_n e^{-(\phi_n + \phi_k)}}{\lambda_k + \lambda_n} = h_k^2 e^{-\phi_k}.
\]

(23)

By Cramer’s rule, one can show that solutions to the linear system of equations (23) are

\[
g_k = -e^{\phi_k} \frac{T_k}{\tau}
\]

(24)

where

\[
\tau = \det |A|
\]

(25)

with a \( N \times N \) matrix \( A \)

\[
A = \left[ \delta_{kn} + h_k^2 \frac{e^{-(\phi_k + \phi_n)}}{\lambda_k + \lambda_n} \right]_{k,n}
\]

(26)

\( \tau \) is the determinant of a matrix \( A \) with each element \([\cdot]_{k,n} \) at \( k \) th row and \( n \) th column. \( \tau_k \) is the determinant of a matrix \( A \) whose \( k \) th column is replaced by \( \frac{\partial}{\partial x}[\cdot]_{k,..} \).

Remark 1. The equation (11) is the Fredholm integral equation of the second kind with the degenerate kernel and has a solution. If its solution has zero asymptotic behavior as \( \lambda \to \infty \) and the equation (11) takes the homogenous integral equation (its solution is normalized to zero), then there exists only zero solution \( M \equiv 0 \) because of the determinant \( \tau \neq 0 \) with an appropriate choice of \( h_k \ (k = 1, \ldots, N) \).
Remark 2. We have less restriction on locations of eigenvalues $\lambda_k, k = 1, \ldots, N$, in comparison with the classical inverse scattering method where, in general, eigenvalues are assumed to be on the positive imaginary axis. In our context, $N$-soliton solution appears from eigenvalues on the whole real line. We can demand $h_k^2 \to -h_k^2$ for negative eigenvalues $\lambda_k$ to avoid singularity of a matrix $A$.

4. Compatibility Condition

We introduce two linear differential operators that are crucial to recover the potential $u(x,t)$. So far, we have not discussed any connection between a function $M(\lambda, \bar{\lambda})$ and a potential $u(x,t)$. One can choose operators $L_1$ and $L_2$

\begin{align*}
L_1 &= D_1^2 - \lambda^2 - u, \\
L_2 &= D_2 + 4D_1^3 + 6uD_1 + 3u_x,
\end{align*}

with $D_1 = \frac{\partial}{\partial x} + \lambda$ and $D_2 = \frac{\partial}{\partial t} - 4\lambda^3$.

These are similar to our familiar compatibility operators of the KdV equation except that $D_1$ and $D_2$ are not $\partial_x$ and $\partial_t$. It is important to note that $L_1$ and $L_2$ commute with $\bar{\partial}$,

\begin{equation}
L_i\bar{\partial} = \bar{\partial}L_i \quad i = 1, 2.
\end{equation}

By applying $L_i$ to the equation (10)

\begin{equation}
L_i(\partial M(\lambda, \bar{\lambda})) = L_i(T_0(\lambda, \bar{\lambda})e^{-2\lambda x+8\lambda^3 t}M(-\lambda, -\bar{\lambda}))
\end{equation}

and by commutativity (29) and $L_ie^{-2\lambda x+8\lambda^3 t}M = e^{-2\lambda x+8\lambda^3 t}L_iM$

\begin{equation}
\bar{\partial}(L_iM(\lambda, \bar{\lambda})) = T_0(\lambda, \bar{\lambda})e^{-2\lambda x+8\lambda^3 t}L_iM(-\lambda, -\bar{\lambda}).
\end{equation}

By simple operations, we verify that $L_iM$ solves the $\bar{\partial}$-equation (10).

We apply $L_1$ and $L_2$ to $M$

\begin{align*}
L_1 M &= \frac{\partial^2 M}{\partial x^2} + 2\lambda \frac{\partial M}{\partial x} + uM \quad (32) \\
L_2 M &= \frac{\partial M}{\partial t} + 4 \frac{\partial^3 M}{\partial x^3} + 12\lambda \frac{\partial^2 M}{\partial x^2} + 12\lambda^2 \frac{\partial M}{\partial x} + 6u \left( \frac{\partial M}{\partial x} + \lambda M \right) + 3u_x M \quad (33)
\end{align*}
and substitute an asymptotic expansion around $\lambda = \infty$, $M = 1 + M_1/\lambda + M_2/\lambda^2 + \cdots$, into the equation (32) as $\lambda \to \infty$

$$L_1 M \to 2 \frac{\partial M_1}{\partial x} + u. \quad (34)$$

We impose the condition $M_1 = -\frac{\int_{-\infty}^{x} u dx}{2} = -\frac{1}{2} \partial^{-1} u$ so that

$$L_1 M \to 0 \quad (35)$$

as $\lambda \to 0$.

From the equations (11), it implies that $L_1 M$ is a solution to the homogeneous integral equation because of an asymptotic behavior $L_1 M \to 0$ as $\lambda \to 0$. As stated in remark 1, this gives a rise to the zero solution. Thus, for all $\lambda \in \mathbb{R}$,

$$L_1 M \equiv 0. \quad (36)$$

By knowing $L_1 M \equiv 0$, one can find $F_2$ in asymptotic expansion of $M$ around $\lambda = \infty$. Then by the same way, it can be easily shown that

$$L_2 M \to 0 \quad (37)$$

as $\lambda \to 0$ which yields

$$L_2 M \equiv 0 \quad (38)$$

for all $\lambda \in \mathbb{R}$.

From two linear equations

$$\frac{\partial^2 M}{\partial x^2} + 2\lambda \frac{\partial M}{\partial x} + u M = 0, \quad (39)$$

$$\frac{\partial M}{\partial t} + 4 \frac{\partial^3 M}{\partial x^3} + 12\lambda \frac{\partial^2 M}{\partial x^2} + 12\lambda^2 \frac{\partial M}{\partial x} + 6u \left( \frac{\partial M}{\partial x} + \lambda M \right) + 3u_x M = 0, \quad (40)$$

one can derive a simple equation

$$M_t + M_{xxx} + 3uM_x = 0 \quad (41)$$

from which, by substituting $M = 1 + \frac{M_1}{\lambda} + \frac{M_2}{\lambda^2} + \cdots$ and considering only the term of $1/\lambda$

$$M_{1t} + M_{1xxx} + 3uM_{1x} = 0, \quad (42)$$
and since $M_1 = -\frac{1}{2} \partial^{-1} u$, one obtains
\begin{align}
\partial^{-1} u_t + u_{xx} + 3u^2 &= 0, 
\tag{43} \\
u_t + 6uu_x + u_{xxx} &= 0. 
\tag{44}
\end{align}

The term $M_1$ indeed derives the KdV from two linear equations, independent of the value $\lambda$.

To summarize, we first introduced a function $M$ that solves $\partial$-problem (11) with the choice of the degenerate kernel (13) that leads to the system of equations (23). We introduced two linear operators (32-33) and determined asymptotic series of $M$ around $\lambda = \infty$ such that $L_i M \to 0$ as $\lambda \to \infty$ ($i = 1, 2$). Since a function applied with operators, $L_i M$ ($i = 1, 2$), are also solutions of the $\partial$-equation (10) with the kernel (13), by virtue of unique solvability, it gives a rise to zero solution, $L_i M = 0$ ($i = 1, 2$). Furthermore, these linear operators derive the KdV equation from the first term $M_1$ in asymptotic series of $M$.

By using the system of equations (23), we can solve for $M_1$, as described in the next section.

5. \textit{N}-Soliton Solutions

Here we derive $N$-soliton solutions, and the determinant (25) is rather expressed by the Hirota’s determinant as necessary for our formulating symmetric $N$-soliton solutions.

By summing over an index $k = 1, \ldots, N$ of the equation (24), we obtain $\sum_{k=1}^{N} g_k e^{-\phi_k} = -\frac{1}{\tau} \sum_{k=1}^{N} \tau_k = -\frac{\partial}{\partial x} \ln \tau$. With $\sum_{k=1}^{N} f_k = \sum_{k=1}^{N} g_k e^{-\phi_k}$, we can write
\begin{equation}
\sum_{k=1}^{N} f_k = -\frac{\partial}{\partial x} \ln \tau. \tag{45}
\end{equation}

With the equation (20) and an asymptotic series of $M$ around $\lambda = \infty$, it is easy to see
\begin{align}
M_1 &= \sum_{k=1}^{N} f_k 
\tag{46} \\
&= -\frac{\partial}{\partial x} \ln \tau \tag{47}
\end{align}

and since $M_1 = -\frac{1}{2} \partial^{-1} u$
\begin{equation}
u = 2 \frac{\partial^2}{\partial x^2} \ln \tau \tag{48}
\end{equation}
that is N-solitonic solution of the KdV equation.

The determinant $\tau$ (25) can take an alternative form that was formulated by Hirota [4]

$$
\tau = \sum_{k=1}^{2N} \prod_{i>j} \left( \frac{\eta_i - \eta_j}{\eta_i + \eta_j} \right)^2 \exp \left( \sum_j \xi_j \right),
$$

(49)

with

$$
\xi_i = -2\eta_i(x - x_{0i} - 4\eta_i^2 t).
$$

(50)

$\eta_i$ and $x_{0i}$ are an eigenvalue and initial solitonic position, and notice that $x_{0i} = \frac{1}{2\eta_i} \ln \frac{h_i^2}{2\eta_i}$. (51)

A set $\sigma_k$ contains elements, selected out of 1, 2, ..., $N$. The sum $\sum_{k=1}^{2N}$ sums all possible elements in a set $\sigma_k$. For instance, when $N = 3$, $\sigma_1 = \emptyset$, $\sigma_2 = \{1\}$, $\sigma_3 = \{2\}$, $\sigma_4 = \{3\}$, $\sigma_5 = \{1, 2\}$, $\sigma_6 = \{1, 3\}$, $\sigma_7 = \{2, 3\}$, $\sigma_8 = \{1, 2, 3\}$. We define that $\sigma_1$ contains no element. There are $2^3$ combinations. The product $\prod_{i > j}^{\sigma_k}$ runs all possible pairs in a set $\sigma_k$ with $i > j$. The sum $\sum_{j}^{\sigma_k}$ sums $j$ of each element in a set $\sigma_k$. The Hirota’s determinant for 3-soliton solutions, for example to clarify the above formula, are given by

$$
\tau = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + \left( \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right)^2 e^{\xi_1+\xi_2} + \left( \frac{\eta_1 - \eta_3}{\eta_1 + \eta_3} \right)^2 e^{\xi_1+\xi_3} + \left( \frac{\eta_2 - \eta_3}{\eta_2 + \eta_3} \right)^2 e^{\xi_2+\xi_3} + \left( \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right)^2 \left( \frac{\eta_1 - \eta_3}{\eta_1 + \eta_3} \right)^2 \left( \frac{\eta_2 - \eta_3}{\eta_2 + \eta_3} \right)^2 e^{\xi_1+\xi_2+\xi_3}.
$$

(52)

**Remark 3.** N-soliton solutions are invariant under multiplying the determinant $\tau$ by $Ce^\xi$, where $\xi = -2\lambda(x - x_0 - 4\lambda^2 t)$ and $C$ is constant.

Let $\tilde{\tau} = \tau Ce^\xi$ so that

$$
-2 \frac{\partial^2}{\partial x^2} \ln \tilde{\tau} = -2 \frac{\partial^2}{\partial x^2} \ln \tau Ce^{-2\lambda(x-x_0-4\lambda^2 t)}
$$

= $-2 \frac{\partial^2}{\partial x^2} [\ln \tau + \ln C e^{-2\lambda(x-x_0-4\lambda^2 t)}]$

$$
= -2 \frac{\partial^2}{\partial x^2} \ln \tau = u.
$$

We call $\tilde{\tau} = \Delta Ce^\xi$ the equivalent determinant.
6. Symmetric $N$-Soliton Solutions at an initial time $t = 0$

An idea of obtaining symmetric $N$-soliton solutions is rather simple, stemming from that the determinant $\tau$, being an even function, may yield an even function of $u$. To do that, we introduce the following changes of parameters

$$e^{2\eta_1x_0} = \prod_{i \neq k} \frac{\eta_i + \eta_k}{\eta_i - \eta_k}. \quad (54)$$

The product runs all possible pairs for $i$ except $i \neq k$. We denote $q_i = e^{2\eta_i x_0}$. For instance, when $N = 3$, $q_1 = \frac{\eta_1 + \eta_2}{|\eta_1 - \eta_2| |\eta_2 - \eta_3|}$. Furthermore, by multiplying the Hirota’s determinant by $\frac{1}{2} e^{(\eta_1 + \cdots + \eta_N)x}$ and grouping terms that share same coefficients, one can derive

$$\bar{\tau} = \cosh(\eta_1 + \cdots + \eta_N)x + q_1 \cosh(-\eta_1 + \cdots + \eta_N)x + \cdots + q_N \cosh(\eta_1 + \cdots - \eta_N)x +$$

$$+ \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 q_1 q_2 \cosh(-\eta_1 - \eta_2 + \eta_3 \cdots + \eta_N)x + \cdots. \quad (55)$$

Grouping two coefficients of exponential functions yields cosine hyperbolic functions. Exponential functions of terms, corresponding with $\sigma_k$ and $\{1, 2, \ldots, N\} \setminus \sigma_k$, have common coefficients, so that they can be grouped.

One may express, with $t = 0$,

$$\bar{\tau} = \sum_{k=1}^{2N-1} \prod_{i > j} (\eta_i - \eta_j)^2 \prod_{i>j} q_i \cosh \left(\sum_{i=1}^{N} \eta_i - 2 \sum_{j} \eta_j\right) \quad (56)$$

A set $\bar{\sigma}_k$ contains elements, selected out of $1, 2, \ldots, N$. There are a fewer combinations in $\bar{\sigma}_k$ than in $\sigma_k$ of the equation (49). There are $2^{N-1}$ combinations in total. Two combinations $\sigma_k$ and $\{1, 2, \ldots, N\} \setminus \sigma_k$ are equivalent for a set $\bar{\sigma}_k$, since terms corresponding with $\sigma_k$ and $\{1, 2, \ldots, N\} \setminus \sigma_k$ are put together to get a cosine hyperbolic function. The product $\prod_{i > j} \bar{\sigma}_k$ runs all possible pairs in a set $\bar{\sigma}_k$ with $i > j$. The sum $\sum_{j} \bar{\sigma}_k$ sums $j$ of each element in a set $\bar{\sigma}_k$.

For instance, when $N = 3$, we have $\bar{\sigma}_1 = \emptyset, \bar{\sigma}_2 = \{1\}, \bar{\sigma}_3 = \{2\}, \bar{\sigma}_4 = \{3\},$

$$\bar{\tau} = \cosh(\eta_1 + \eta_2 + \eta_3)x + q_1 \cosh(-\eta_1 + \eta_2 + \eta_3)x + q_2 \cosh(\eta_1 - \eta_2 + \eta_3)x + q_3 \cosh(\eta_1 + \eta_2 - \eta_3)x. \quad (57)$$

**Remark 4.** By changes of parameters (54), the equivalent determinant $\bar{\tau} > 0$ appears to be an even function, which gives an even solution $u$ of the KdV soliton. We
call the KdV solutions by the equivalent determinant $\tilde{\tau}$ symmetric $N$-soliton solutions.

Locations and heights of solitonic peaks in symmetric $N$-soliton solutions are solely influenced by locations of eigenvalues. We must study how symmetric $N$-soliton solutions vary, depending on various spacing within the interval of the maximum and minimum eigenvalues $\eta_{\text{max}}$ and $\eta_{\text{min}}$. Before moving on to the next section, let us see the following figures.

![Symmetric 7-Soliton Solutions](image)

- (a) Equally distributed eigenvalues: $\eta_{\text{max}} = 1 + \frac{1}{10}, \eta_{\text{min}} = 1 - \frac{1}{10}$
- (b) $\eta_k = 1 + \frac{1}{10} \cos(k\pi/6)$, $k = 0, 1, \ldots, 6$
- (c) Eigenvalues concentrated at the center: $\eta_k = 1, 1 \pm \frac{1}{7 \times 10}, 1 \pm \frac{1}{5 \times 10}, 1 \pm \frac{1}{10}$
- (d) Eigenvalues concentrated at the sides: $\eta_k = 1, 1 \pm \frac{3}{7 \times 10}, 1 \pm \frac{6}{7 \times 10}, 1 \pm \frac{1}{10}$
- (e) Eigenvalues concentrated at the center and sides: $\eta_k = 1, 1 \pm \frac{1}{7 \times 10}, 1 \pm \frac{6}{7 \times 10}, 1 \pm \frac{1}{10}$
- (f) $\eta_k = 1 \pm a, 1 \pm 3a, 1 \pm 6a$, where $a = \frac{1}{60}$

Figure 1: Symmetric 7-Soliton Solutions (56)

The above Figures show symmetric 7-soliton solutions with basic variations of
eigenvalues’ spacing. One may ask, is it possible to find spacing of eigenvalues that leads to equidistant solitonic peaks with same height for $N$-soliton solutions? We set this to be our open problem. In the next section, we derive an asymptotic form of symmetric $N$-soliton solutions with equidistant eigenvalues, which might have a meaningful result as $N \to \infty$. We must, however, emphasize that this choice of eigenvalues’ spacing is based on heuristic observation.

7. Symmetric Soliton Solutions for Equally Spaced Eigenvalues

For symmetric $N$-soliton solutions (56), we choose eigenvalues to be

$$\eta_k = \eta + (k - \frac{N+1}{2})\delta_N, \quad k = 1, 2, \ldots, N.$$  \hspace{1cm} (58)

We consider only an odd number of $N$. $\eta$ is some positive constant, and $\delta_N$ is dependent on $N$. $\delta_N$ is chosen in such a way that $\eta_{\text{max}} = \max_k \{\eta_k\}$ and $\eta_{\text{min}} = \min_k \{\eta_k\}$ stay the same for any odd integer $N > 0$.

**Formula 1.** An asymptotic solution $v$ of a symmetric $N$-soliton solution is given by

$$v = -2\frac{\partial^2}{\partial^2 x} \log \tau_a$$  \hspace{1cm} (59)

with

$$\tau_a = \sum_{k=0}^{N-1} A_{k,N} \left( \frac{2\eta}{\delta_N} \right)^{k(N-k)} \cosh((N-2k)\eta x),$$  \hspace{1cm} (60)

where

$$A_{k,N} = \sum_{n_1=1}^{N-k+1} \sum_{n_2=n_1+1}^{N-k+2} \cdots \sum_{n_{k-1}=n_{k-2}+1}^{N-1} \sum_{n_k=n_{k-1}+1}^{N} \frac{\prod_{i<j}^{k}(n_i - n_j)^2}{\prod_{i=1}^{k}(N-n_i)!(n_i - 1)!},$$  \hspace{1cm} (61)

such that for a sufficiently small $\delta_N > 0$

$$v \sim u,$$  \hspace{1cm} (62)

where $u$ is given in the equation (48) with the equivalent determinant (56).

One may choose $\delta_N = \frac{2c}{N-1}$ so that $\eta_{\text{max}} = \eta + c$ and $\eta_{\text{min}} = \eta - c$, where $\eta$ and $c$ are positive real constants. To derive the above formula, it requires a lengthy and
meticulous computation. It would be possible to express a coefficient $A_{k, N}$ in a sim-
pler form by grouping identical values in sums and finding patterns for simplification.

Brief derivations of the formula are as follows.
For a sufficiently small $\delta_N$, we can express asymptotically

$$q_l \sim \frac{1}{(N-l)! (l-1)!} \left( \frac{2 \eta}{\delta_N} \right)^{N-1},$$  \hfill (63)

$$\left( \frac{\eta_l - \eta_k}{\eta_l + \eta_k} \right)^2 \sim (l-k)^2 \left( \frac{\delta_N}{2 \eta} \right)^2,$$  \hfill (64)

and

$$\cosh \left( \sum_{i=1}^N \eta_i - 2 \sum_j \eta_j \right) x \sim \cosh((N-2l)\eta x).$$  \hfill (65)

The integer $l$ in $\cosh((N-2l)\eta x)$ means the number of elements in $\tilde{\sigma}_k$. Then, the
asymptotic form of symmetric $N$-soliton solutions can take form of the series

$$\tilde{\tau} \sim \sum_{k=0}^{N-1} A_{k, N} \left( \frac{2 \eta}{\delta_N} \right)^{k(N-k)} \cosh((N-2k)\eta x),$$  \hfill (66)

where coefficients $A_{k, N}$ depend on $N$ and $k$.
Coefficients $A_{k, N}$ can be written as

$$A_{0, N} = 1,$$  \hfill (67)

$$A_{1, N} = \sum_{k=1}^N \frac{1}{(N-k)! (k-1)!},$$  \hfill (68)

$$A_{2, N} = \sum_{k=1}^{N-1} \sum_{l=k+1}^N \frac{(k-l)^2}{(N-k)! (k-1)! (N-l)! (l-1)!},$$  \hfill (69)

$$A_{5, N} = \sum_{k=1}^{N-4} \sum_{l=k+1}^{N-3} \sum_{m=l+1}^{N-2} \sum_{n=m+1}^{N-1} \sum_{j=m+1}^N \frac{(k-l)^2 (k-m)^2 (k-n)^2 (k-j)^2 (l-m)^2}{(N-k)! (k-1)! (N-l)! (l-1)! (N-m)! (m-1)!} \times$$

$$\times \frac{(l-n)^2 (l-j)^2 (m-n)^2 (m-j)^2 (n-j)^2}{(N-n)! (n-1)! (N-j)! (j-1)!},$$  \hfill (70)
or more generally

$$A_{k,N} = \sum_{n_1=1}^{N-k+1} \sum_{n_2=n_1+1}^{N-k+2} \ldots \sum_{n_{k-1}=n_{k-2}+1}^{N-1} \sum_{n_k=n_{k-1}+1}^{N} \frac{\prod_{i<j}^{(k)}(n_i - n_j)^2}{\prod_{i=1}^{k} (N - n_i)!(n_i - 1)!},$$  \hspace{1cm} (71)

where the product $\prod_{i<j}^{(k)}$ runs all possible pairs of $i, j$ out of $\{1, 2, \ldots, k\}$ with $i < j$.

8. Numerical Experiments

Numerical implementation of an asymptotic form (60) can be realized more simply than of an analytical form (56). By using an asymptotic form (60), we want to make observation whether symmetric $N$-soliton solutions with equidistant eigenvalues can approach periodic solutions as $N$ increases. In this section, there is no rigorous analysis involved in our approach, but interesting observations are to be made.
We set $\eta = \frac{1}{3}$ and $\delta_N = \frac{0.002}{N-1}$ so that $\eta_{\text{max}} = \frac{1}{3} + 0.001$ and $\eta_{\text{min}} = \frac{1}{3} - 0.001$. The width of the band is 0.002 which is relatively narrow.
Figure 2: Symmetric Soliton Solutions from the equation (60)
We found locations of solitonic peaks numerically. We show only positive locations of solitonic peaks because of solution’s symmetry, and location at zero is excluded in the following list.

3-solitons: 20.5466
5-solitons: 20.9781, 42.5644
7-solitons: 21.1548, 42.5831, 64.7776
9-solitons: 21.2516, 42.6612, 64.4478, 87.0739
11-solitons: 21.3128, 42.7291, 64.3767, 86.4558, 109.417
13-solitons: 21.3551, 42.7834, 64.3697, 86.2295, 108.554, 131.789
15-solitons: 21.3860, 42.8266, 64.3826, 86.1303, 108.179, 130.716, 154.181, ...
17-solitons: 21.4096, 42.8615, 64.4017, 86.0848, 107.983, 130.201, 152.924, ...
19-solitons: 21.4283, 42.8903, 64.4220, 86.0649, 107.870, 129.907, 152.278, ...
21-solitons: 21.4433, 42.9142, 64.4417, 86.0583, 107.803, 129.724, 151.888, ...

Distances between adjacent solitonic peaks are the following

3-solitons: 20.5466
5-solitons: 20.9781, 21.5863
7-solitons: 21.1548, 21.4283, 22.1945

It can be observed that, as the number of $N$ increases, each distance between peak’s locations seems to converge to its limit. What is interesting is that distances in the first two columns are increasing sequences as $N$ increases, and distances from the third or greater columns are decreasing sequences as $N$ increases.

Depending on choice of eigenvalues, one can get symmetric $N$-solitons that are partially emerged with each other. One can, for example, pick values, $\eta = \frac{1}{3}$ and
\( \delta_N = \frac{0.2}{N-1} \) which is a wider band than the last example. Eigenvalues are distributed equally in the interval of \( \frac{1}{3} - 0.1 \leq \eta_k \leq \frac{1}{3} + 0.1 \).

![Graphs showing symmetric soliton solutions](image)

Figure 3: Symmetric Soliton Solutions from the equation (68)

For both Figures 2 and 3, additional solitons at each increment of odd number \( N \) appear on the furtherest both sides of a preceding solution. Locations and heights of solitonic peaks seem to approach being uniform. The question whether or not limits of symmetric \( N \)-soliton solution (56) or asymptotic solution (59) as \( N \to \infty \) can lead to the cnoidal waves is beyond our cope. Furthermore, equidistant spacing in eigenvalues would not be a only promising way of filling a finite gap. Having all said, we introduced symmetric \( N \)-soliton solutions and their asymptotic forms with equidistant eigenvalues that we yet believe are new formulations of solutions and also that show a possible connection to the cnoidal waves as their limits.

9. Acknowledgements

Many thanks to Vladimir Zakharov for very helpful discussions. During the lecture on soliton theory in March 2012, he introduced the change of parameters (54) to derive symmetric \( N \)-soliton solutions and emphasized on the possibility of
obtaining finite gap solutions from symmetric $N$-soliton solutions with appropriate choice of spacing in eigenvalues. That was my inspiration to start working on the paper.

References


